*Normal Congruences Determined by Centers of Geodesic Curvature.**

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Introduction.

In 1892 Caronnet announced the following theorem: † The necessary and sufficient condition that the lines which join the centers of geodesic curvature of the curves of an orthogonal system on a surface shall form a normal congruence, is that the corresponding radii of geodesic curvature be functions of one another or that the curves in one family or both have constant geodesic curvature. The present article is concerned with the determination of normal congruences of this kind and with an investigation of their properties. For the sake of brevity we shall refer to them as *associated normal congruences.*

In § 1, the theorem of Caronnet is proved and is extended to the case where one family of the orthogonal system is composed of geodesics, *i. e.*, when one of the radii of geodesic curvature is constantly infinite. In establishing the existence of associated normal congruences, there are three cases to be investigated: 1) the geodesic curvature is the same constant for the curves of each of the two families forming the orthogonal system; 2) it is constant and the same for the curves of only one family, and 3) the radii of geodesic curvature are functions of one another. The case in which both radii of geodesic curvature are given constants is discussed in §§ 2, 3 and 4. It is there shown that the surface on which such a system of curves exists must be pseudo-spherical and of total curvature equal to the negative of the sum of the squares of the constant geodesic curvatures. The associated normal congruence is found to be a congruence of normals of a Bianchi transform of the given surface. It is also proved that each of these congruences is determined by a single infinity of orthogonal systems or a double infinity of systems whose curves intersect under a constant angle and have constant geodesic curvature.

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† "Sur les centres de courbure géodésique," *Comptes Rendus,* Vol. CXV, p. 589–592.

The general discussion of the subject occurs in §§ 5, 6, 7 and 8. It is shown in § 5 that associated normal congruences exist for any surface when the radii of geodesic curvature are given analytic functions of one another or when one radius of geodesic curvature is a constant. Characteristic properties of these congruences are discovered in §§ 6–7. The condition that a general normal congruence whose lines lie in the tangent planes of the given surface be an associated normal congruence is obtained. It is found that every system whose curves intersect under a constant angle, and for which the corresponding radii of geodesic curvature are functions of one another or for which one radius is a constant, determines an associated normal congruence. Further, the problem of determining these new systems is found to be the same as that of determining the orthogonal system. The restriction in the theorem of Caronnet that the curves be orthogonal may then be replaced by the condition that they intersect under a constant angle, but the more general theorem so obtained will not give any new congruences. In § 8, the condition that the developables of the associated normal congruence correspond to the lines of curvature on the given surface is investigated. The remainder of the paper deals with problems peculiar to associated normal congruences, determined by orthogonal systems, the curves of one family of which have a constant geodesic curvature. In the discussion we are led to consider surfaces on which one family of the lines of curvature have a constant geodesic curvature. Except in § 8 and most of § 9, the results obtained express deformation properties.

§ 1. *Theorem of Caronnet.*

We will give, for reference, a proof of the theorem of Caronnet. Let S be a surface referred to an orthogonal system of parametric curves $u=$ const., $v=$ const. Associate with S a set of moving axes, or trihedral, whose x- and y-axes are tangent to $v=$ const. and $u=$ const., respectively, and whose z-axis is normal to the surface. From the formula for the moving trihedral, this choice gives *

$$\xi_1=\eta=0, \quad r=-\frac{1}{\eta_1}\frac{\partial \xi}{\partial v}, \quad r_1=\frac{1}{\xi}\frac{\partial \eta_1}{\partial u}.$$

Hence the radii of geodesic curvature R_1 and R_2 of the curves $v=$ const. and $u=$ const. respectively are given by the formulæ

$$R_1=\xi/r \text{ and } R_2=\eta_1/r_1.$$

The coordinates of the centers of geodesic curvature of the parametric curves,

* For notation used in this paper see Eisenhart's "Differential Geometry," §§ 57, 68–72, 76 and 119.

with respect to the moving trihedral, are $(0, R_1, 0)$ and $(-R_2, 0, 0)$. Consequently the coordinates of any point P on the line L joining these centers are $(R_2(\lambda - 1),\ \lambda R_1, 0)$, where $\lambda = \dfrac{t}{\sqrt{R_1^2 + R_2^2}}$ and t is the distance from $(-R_2, 0, 0)$ to P.

As M, the vertex of the moving trihedral, receives an infinitesimal displacement along the surface, the point P receives a displacement whose x and y components are:

$$\delta x = d[(\lambda - 1) R_2] + \xi\, du - \lambda R_1 (r\, du + r_1\, dv),$$
$$\delta y = d(\lambda R_1) + \eta_1\, dv + (\lambda - 1) R_2 (r\, du + r_1\, dv).$$

The necessary and sufficient condition that the line L generate a normal congruence is that there exist a point P on L such that for displacements of M in all directions on the surface the displacements of P are perpendicular to L, *i. e.*,

$$R_2\, \delta x + R_1\, \delta y + 0\, \delta z = 0. \tag{1}$$

This condition reduces to

$$R_2\, dR_2 = d[\lambda (R_1^2 + R_2^2)] + (R_1^2 + R_2^2)\, d\lambda.$$

Upon substitution of the value of λ this becomes

$$R_2\, dR_2 = (R_1^2 + R_2^2)^{1/2}\, dt. \tag{2}$$

Unless R_2 is a constant and consequently t also, it follows from (2) that $\dfrac{\partial(t, R_2)}{\partial(u, v)} = 0$, *i. e.*, t is a function of R_2, and hence by (2) that R_1 is a function of R_2. Conversely, if in (2) R_1 is assumed to be a function of R_2, t is so defined that the congruence will be normal. When R_2 is a constant, t is also a constant. Hence, if one of the radii of geodesic curvature is a constant, the corresponding center of geodesic curvature generates an orthogonal surface of the associated normal congruence.

The establishment of the existence for any surface of systems of curves of the kind required, and hence of the existence of associated normal congruences, will be deferred until later. Let us now consider an exceptional case, which has not been included in the preceding discussion; namely, when one of the radii is infinite. This happens when one of the families of the orthogonal system is composed of geodesics. Such systems are known to exist on any surface. If $R_2 = \infty$, the direction cosines of the line of the congruence become $1, 0, 0$, and the coordinates of the point P on L, $(t, R_1, 0)$. The condition (1) that P generates an orthogonal surface of the congruence is satisfied by $t = \text{const.}$ The congruence is therefore normal. In particular, the point $(0, R_1, 0)$ generates an orthogonal surface of the congruence which is complementary to the given surface for the geodesic family.

§ 2. *Geodesic Curvature Constant in Both Families. Pseudospherical Surfaces.*

Let S be a surface whose parametric curves form an orthogonal system, and assume that the geodesic curvatures of $u = \text{const.}$ and $v = \text{const.}$ are a and b respectively. Then, if the linear element is of the form $ds^2 = E\,du^2 + G\,dv^2$, it follows that

$$a = \frac{1}{\sqrt{EG}} \frac{\partial \sqrt{G}}{\partial u}, \qquad b = -\frac{1}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial v}. \tag{3}$$

Differentiating these with respect to v and u respectively and adding the resulting equations, we obtain

$$\frac{\partial^2}{\partial u\, \partial v} \log \frac{E}{G} - 0.$$

Therefore E/G is equal to the ratio of a function of u and a function of v. The system is thus isothermic and the linear element of the surface can be put in the form $ds^2 = \lambda^2 (du^2 + dv^2)$. Formulæ (3) under these conditions become

$$a = \frac{1}{\lambda^2} \frac{\partial \lambda}{\partial u}, \qquad b = -\frac{1}{\lambda^2} \frac{\partial \lambda}{\partial v}.$$

Integrating these we see that $\lambda = (bv - au)^{-1}$ and that the linear element of the surface is [*]

$$ds^2 = \frac{du^2 + dv^2}{(bv - au)^2}. \tag{4}$$

The total curvature of this quadratic form is $K = -(a^2 + b^2)$. Hence every surface with this linear element is pseudospherical. Since all pseudospherical surfaces with the same total curvature are applicable on one another, and since the geodesic curvature of any curve on a surface is unchanged by deformation of the surface, there exists on every pseudospherical surface at least one orthogonal system for which, when parametric, the surface has a linear element of the form (4).

Let us inquire whether there is more than one such system on a pseudospherical surface, assuming in the discussion that the surface is referred to some orthogonal system. Then any other orthogonal system is determined by

$$\frac{dv}{du} = \frac{\xi}{\eta_1} \tan \theta, \tag{5}$$

$$\frac{dv}{du} = -\frac{\xi}{\eta_1} \cot \theta, \tag{6}$$

where θ is the angle the curves given by (5) make with $v = \text{const.}$ If the curves

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[*] Bianchi, "Lezioni Di Geometria Differenziale," second ed., Vol. I, p. 211.

whose differential equations are (5) and (6) have geodesic curvatures b and a respectively, then

$$b\,\xi\,\eta_1 = \frac{\partial}{\partial u}(\eta_1 \sin\theta) - \frac{\partial}{d v}(\xi\cos\theta),$$

$$a\,\xi\,\eta_1 = \frac{\partial}{\partial u}(\eta_1 \cos\theta) + \frac{\partial}{\partial v}(\xi\sin\theta).$$

Solving these for θ_u and θ_v, we have

$$\theta_u = \frac{\xi_v}{\eta_1} + \xi\,(b\cos\theta - a\sin\theta), \tag{7}$$

$$\theta_v = -\frac{\eta_{1u}}{\xi} + \eta_1\,(a\cos\theta + b\sin\theta). \tag{8}$$

The condition of integrability $\theta_{uv} = \theta_{vu}$ reduces to

$$a^2 + b^2 = \frac{r_{1u} - r_v}{\xi\,\eta_1} = -K.$$

Hence the following theorem:

THEOREM 1. *There exists on a pseudospherical surface of total curvature* $-(a^2 + b^2)$ *a single infinity of orthogonal systems whose families have geodesic curvatures a and b.*

If a and b are replaced by other constants c and d, the condition of integrability requires only that $c^2 + d^2 = a^2 + b^2$. Hence we have

THEOREM 2. *There exists on any pseudospherical surface a double infinity of systems of orthogonal curves such that the curves in each family have the same constant geodesic curvature.*

The substitution $t = \tan\theta/2$ will reduce equations (7) and (8) to a Riccati equation, and we have

THEOREM 3. *Given any orthogonal system of curves on a pseudospherical surface, the determination of all orthogonal systems, the two families of which have given constant geodesic curvatures, requires the solution of a Riccati equation.*

§ 3. *Transformations of Bianchi.*

Let us return to the study of a pseudospherical surface with the linear element (4) and of the associated normal congruence formed by joining corresponding centers of geodesic curvature. If M denotes a point on the surface, L the corresponding line of the congruence, and P_1 the foot of the perpendicular from M to L, we wish to prove that P_1 generates an orthogonal surface of the congruence, that this surface is pseudospherical and of the same total curvature

as the original surface, and finally that the lines of curvature on both surfaces correspond. The distances from M to the two centers of geodesic curvature of the parametric curves are constant; hence the distance from either of these centers to P_1 is constant. But in § 1 it was shown that a point at a constant distance on L from the center of constant geodesic curvature generates an orthogonal surface. Hence P_1 generates such a surface.

In order to obtain the total curvature of this surface, we will find the product of the focal distances, measured from P_1, of the congruence of lines L. If a and b are the geodesic curvatures of $u = \text{const.}$ and $v = \text{const.}$ respectively, the coordinates of any point P on L will be

$$\left(\frac{-a}{a^2+b^2}+\frac{b\,t}{\sqrt{a^2+b^2}},\quad \frac{b}{a^2+b^2}+\frac{a\,t}{\sqrt{a^2+b^2}},\quad 0\right),$$

where t is the distance from P_1. The displacements of P are

$$\delta x = \frac{b\,dt}{\sqrt{a^2+b^2}}+\frac{du}{bv-au}-(r\,du+r_1\,dv)\left(\frac{b}{a^2+b^2}+\frac{a\,t}{\sqrt{a^2+b^2}}\right),$$

$$\delta y = \frac{a\,dt}{\sqrt{a^2+b^2}}+\frac{dv}{bv-au}+(r\,du+r_1\,dv)\left(\frac{-a}{a^2+b^2}+\frac{b\,t}{\sqrt{a^2+b^2}}\right),$$

$$\delta z = (p\,du+p_1\,dv)\left(\frac{b}{a^2+b^2}+\frac{a\,t}{\sqrt{a^2+b^2}}\right)-(q\,du+q_1\,dv)\left(\frac{-a}{a^2+b^2}+\frac{b\,t}{\sqrt{a^2+b^2}}\right).$$

Since b, a and 0 are proportional to the direction-cosines of the line of the congruence, the conditions that P be displaced in the direction of the line L are

$$\frac{\delta x}{b}=\frac{\delta y}{a}=\frac{\delta z}{0}.$$

These are equivalent to the equations

$$a\,\delta x - b\,\delta y = 0, \qquad \delta z = 0,$$

which may be reduced to the following, when the expressions for δx, δy and δz are substituted:

$$\frac{dv}{du}=\frac{a-b\sqrt{a^2+b^2}\,t}{b+a\sqrt{a^2+b^2}\,t}, \tag{9}$$

$$\frac{dv}{du}=-\frac{p\,b+q\,a+\sqrt{a^2+b^2}\,(p\,a-p\,b)\,t}{p_1\,b+q_1\,a+\sqrt{a^2+b^2}\,(p_1\,a-q_1\,b)\,t}. \tag{10}$$

The elimination of dv/du between these equations gives a quadratic in t whose

roots are the distances from P_1 to the focal points of the congruence. The result of this elimination is the equation

$$\begin{aligned}(a^2+b^2)\,[p\,(a^2-b^2)-a\,b\,(q+p_1)]\,t^2 \\ +\sqrt{a^2+b^2}\,[(p_1+q)\,(a^2-b^2)+4\,a\,b\,p]\,t \\ -[p\,(a^2-b^2)-a\,b\,(q+p_1)]=0,\end{aligned}$$

for which the product of the roots is $-(a^2+b^2)^{-1}$. Therefore the total curvature of the surface generated by P_1 is $-(a^2+b^2)$, the same as the total curvature of the given surface.

To find the equation of the developable surfaces of the congruence let us eliminate t from (9) and (10). The result of this operation is the equation

$$q_1\,d\,v^2+(p_1+q)\,d\,u\,d\,v+p\,d\,u^2=0.$$

As this is the equation of the lines of curvature on the given surface, the developables of the normal congruence, or what amounts to the same thing, the lines of curvature on the orthogonal surfaces of the congruence, correspond to the lines of curvature on the given surface.

Combining these results with the fact that MP_1 is equal to $(a^2+b^2)^{-1/2}$, *i. e.*, equal to $(-K)^{-1/2}$, we have

THEOREM 4. *The point P_1 generates a Bianchi transform of the given surface.*

By virtue of this theorem, any properties which are possessed by Bianchi transforms of the given surface or their congruences of normals are properties of the orthogonal surface generated by P_1 or of the associated normal congruences. One of these properties may be stated thus:

THEOREM 5. *The focal points of this associated normal congruence lie on the tangents to the lines of curvature of the given pseudospherical surface.*

We will now give another proof of this theorem similar to one stated by Darboux.* The tangents to the lines of curvature of an orthogonal surface of a cyclic system pass through the focal points of the associated cyclic congruence. When the circles are in the tangent planes of a pseudospherical surface, with their centers at the points of tangency, their radii are equal to $(-K)^{-1/2}$ and the orthogonal surfaces are Bianchi transforms of the given surface. The associated cyclic congruence is the congruence of normals of the given surface. The relation between a surface and any one of its Bianchi transforms is reciprocal in the sense that either is a Bianchi transform of the other. Therefore the tangents to the lines of curvature on the given surface pass through the focal points of the congruence of normals of the surface generated by P_1.

* Darboux, "Surfaces," Vol. III, p. 430.

§ 4. *Coincidence of Normal Congruences.*

Thus far it has been shown that on any pseudospherical surface of total curvature $-(a^2+b^2)$ there exists a single infinity of orthogonal systems whose families have given constant geodesic curvatures a and b, and that each of these systems determines a congruence of normals of a Bianchi transform of the given surface. It is now our intention to prove that the centers of geodesic curvature of a single infinity of orthogonal systems lie on each of these congruences, or in other words, that each associated normal congruence and each Bianchi transform may be determined by any one of a single infinity of orthogonal systems. If on a surface whose linear element is (4) the geodesic curvatures of an orthogonal system are m and n, then

$$m^2+n^2=a^2+b^2,$$

and if θ denotes the angle which the curves of geodesic curvature n make with $v=$ const., then

$$n=(bv-au)^2\left[\frac{\partial}{\partial u}\frac{\sin\theta}{bv-au}-\frac{\partial}{\partial v}\frac{\cos\theta}{bv-au}\right], \tag{12}$$

$$m=(bv-au)^2\left[\frac{\partial}{\partial u}\frac{\cos\theta}{bv-au}+\frac{\partial}{\partial v}\frac{\sin\theta}{bv-au}\right]. \tag{13}$$

The coordinates of the centers of geodesic curvature of this system of curves, with respect to the moving trihedral, are

$$\left(-\frac{1}{n}\sin\theta,\ \frac{1}{n}\cos\theta,\ 0\right),\quad \left(-\frac{1}{m}\cos\theta,\ -\frac{1}{m}\sin\theta,\ 0\right). \tag{14}$$

If these centers lie on the lines of the congruence determined by the parametric curves, they must satisfy the relation

$$y=ax/b+1/b, \tag{15}$$

which is the equation of the lines with respect to the moving trihedral. Substituting the coordinates of the centers of geodesic curvature (14) in (15), and then solving for $\sin\theta$ and $\cos\theta$, we obtain

$$\cos\theta=\frac{am+bn}{a^2+b^2},\qquad \sin\theta=\frac{an-bm}{a^2+b^2}.$$

These give a constant value for θ which satisfies (12) and (13) and therefore enable us to state the following theorem:

THEOREM 6. *Corresponding curves of any two orthogonal systems whose families have constant geodesic curvature and determines the same normal congruence intersect under a constant angle.*

Since m and n are restricted by the relation (11) only, we have

THEOREM 7. *A single infinity of these systems determines the same associated normal congruence.*

Theorems 6 and 7 show that there exists a double infinity of systems of curves whose families have constant geodesic curvature and intersect under a constant angle, and whose centers of geodesic curvature lie on the lines of the associated normal congruence determined by the parametric curves. If a surface is referred to a system of curves for which g and h are the constant geodesic curvatures of $v =$ const. and $u =$ const. respectively, and ω is the constant angle between these parametric curves, Voss* has shown that the linear element of the surface is

$$d s^2 = \frac{d u^2 + 2 \cos d u\, d v + d v^2}{\left[\frac{(g - h \cos \omega) v + (-h + g \cos \omega) u}{\sin \omega}\right]^2}.$$

The total curvature of this quadratic form is

$$K = \frac{g^2 + h^2 - 2 g h \cos \omega}{\sin^2 \omega},$$

and hence the surface must be pseudospherical. If the x-axis of the moving trihedral is tangent to $v =$ const., it can be shown by the same method as that employed in the case where the curves were orthogonal that the surface generated by P_1 is a Bianchi transform of the given surface. Therefore the congruence determined by any system of curves of this kind must be a congruence of normals of a Bianchi transform of the given surface.

§ 5. *Existence of Associated Normal Congruences for Any Surface.*

In order to prove the existence of associated normal congruences for any surface, assume that an orthogonal system of curves is parametric on the surface and that the curves whose differential equations are (5) and (6) have geodesic curvatures T and T_1 respectively. These geodesic curvatures are then determined by the expressions

$$T = \frac{1}{\xi \eta_1}\left[\frac{\partial}{\partial u}(\eta_1 \sin \theta) - \frac{\partial}{\partial v}(\xi \cos \theta)\right], \tag{18}$$

$$T_1 = \frac{1}{\zeta \eta_1}\left[\frac{\partial}{\partial u}(\eta_1 \cos \theta) + \frac{\partial}{\partial v}(\xi \sin \theta)\right]. \tag{19}$$

Assuming that T_1 is a constant or an analytic function of T, equations (18) and (19) by the elimination of T give a partial differential equation of the

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* *München Sitzungsberichte*, 1906, pp. 247–296.

first order. When T_1 is a constant or equals a constant times T, the differential equation is also of the first degree. Hence the existence of associated normal congruences for any surface is established for cases 2 and 3, and we are enabled to state the following theorem:

THEOREM 8. *Given any analytic functional relation between the corresponding radii of geodesic curvature of the curves of an orthogonal system on a surface, or given that one of these radii is a constant, there exist associated normal congruences and the problem of determining them when any orthogonal system is parametric leads to the solution of a partial differential equation of the first order.*

The results of solving equations (18) and (19) for θ_u and θ_v, when T_1 has been replaced by $f(T)$, are

$$\theta_u = \frac{\xi_v}{\eta_1} + \xi\,(T\cos\theta - f(T)\sin\theta),$$

$$\theta_v = -\frac{\eta_{1u}}{\xi} + \eta_1\,(T\sin\theta + f(T)\cos\theta).$$

Denoting by K the total curvature of the surface and by f' the derivative of $f(T)$ with respect to T, the condition of integrability of these is

$$K + T^2 + f^2 = \frac{T_v}{\eta_1}(\cos\theta - f'\sin\theta) - \frac{T_u}{\xi}(\sin\theta + f'\cos\theta). \tag{20}$$

In order that this equation be independent of θ, so that as in the case of pseudospherical surfaces θ involves a parameter, we must have

$$\begin{aligned} K + T^2 + f^2 &= 0,\\ \xi\,T_v - \eta_1 f'\,T_u &= 0,\\ \xi f'\,T_v + \eta_1 T_u &= 0. \end{aligned}$$

Since the determinant of the system composed of the last two equations is $\xi\eta_1(f'^2 + 1)$, the only real solution is $T_u = 0$, $T_v = 0$, or $T = \text{const.}$ Then by the first condition the surface must be pseudospherical and of total curvature $-(f^2 + T^2)$. Another result arising from the consideration of (20) is that for any set of values T and $f(T)$ there can not exist more than two angles θ.

§ 6. *Some Characteristic Properties of Associated Normal Congruences.*

We now desire to know whether all normal congruences whose lines lie in the tangent planes of a surface are *associated normal congruences.* To carry out the investigation, we assume parametric an orthogonal system of curves on the surface, obtain the condition that a line in the tangent plane generate

a normal congruence and finally derive the condition that this line pass through corresponding centers of geodesic curvature of some orthogonal system. Let ϕ be the angle that the perpendicular from the point of tangency, M, to the line of the congruence makes with the tangent to $v = \text{const.}$, which is taken as the x-axis of an associated moving trihedral, and let h be the length of this perpendicular MP_1. Then the coordinates of any point P on the line will be $(h\cos\phi - t\sin\phi,\ h\sin\phi + t\cos\phi,\ 0)$, where t is the distance from P_1 to P. In order that P move perpendicular to the line, for all infinitesimal displacements of M we must have

$$h\,d\phi + dt - \xi\sin\phi\,du + \eta_1\cos\phi\,dv + (r\,du + r_1\,dv)\,h = 0.$$

This being true for all displacements is equivalent to

$$h\phi_u + t_u - \xi\sin\phi + rh = 0, \tag{21}$$

$$h\phi_v + t_v + \eta_1\cos\phi + r_1 h = 0. \tag{22}$$

The condition of integrability for t of these differential equations is

$$h_u(\phi_v + r_1) - h_v(\phi_u + r) + \eta_{1u}\cos\phi + \xi_v\sin\phi - \eta_1\sin\phi\,\phi_u + \xi\cos\phi\,\phi_v - h\,\xi\,\eta_1 K = 0. \tag{23}$$

Since this does not involve t, any pair of solutions ϕ and h will make (21) and (22) consistent and will therefore determine a normal congruence whose lines lie in the tangent planes of the surface. An inspection of (23) shows that, with certain restrictions, h or ϕ can be chosen arbitrarily and the other determined by a partial differential equation of the first order and first degree.

Denote the radii of geodesic curvature of the two families of the orthogonal system defined by (5) and (6) by R_1 and R_2 respectively. Then the coordinates of the centers of geodesic curvature are $(-R_1\sin\theta,\ R_1\cos\theta,\ 0)$ and $(-R_2\cos\theta,\ -R_2\sin\theta,\ 0)$ respectively. Since the equation of the line of the normal congruence under consideration is

$$x\cos\phi + y\sin\phi = h,$$

the conditions that these two points lie on the line are

$$-\sin\theta\cos\phi + \cos\theta\sin\phi = h/R_1,$$

$$-\cos\theta\cos\phi - \sin\theta\sin\phi = h/R_2,$$

where ϕ and h are solutions of equation (23), and $1/R_1$ and $1/R_2$ are the geodesic curvatures of the curves defined by (5) and (6) respectively. Sub-

stituting for $1/R_1$ and $1/R_2$ their values in terms of ξ, η_1 and θ, and solving for θ_u and θ_v, we have

$$\theta_u = -r + \frac{\xi}{h} \sin\phi, \tag{24}$$

$$\theta_v = -r_1 - \frac{\eta_1}{h} \cos\phi. \tag{25}$$

The condition of integrability of these equations is

$$\eta_{1u} \cos\phi + \xi_v \sin\phi - \eta_1 \sin\phi\, \phi_u + \xi \cos\phi\, \phi_v - h\, \xi\, \eta_1 K$$
$$- \frac{1}{h} (\eta_1 \cos\phi\, h_u + \xi \sin\phi\, h_v) = 0. \tag{26}$$

Since this does not involve θ, any pair of solutions ϕ and h of (23) and (26) will determine a normal congruence made up of the joins of corresponding centers of geodesic curvature of (5) and (6). Further, any set of values ϕ and h, solutions of (23) and (26), will determine by (24) and (25) an infinity of functions θ, which differ only by additive constants, and therefore will give a single infinity of orthogonal system whose centers of geodesic curvature lie on the lines of the normal congruence determined by ϕ and h. Combining this result with Caronnet's theorem, we have the following, a special case of which is given by Theorems 6 and 7.

THEOREM 9. *If the joins of the centers of geodesic curvature of the curves of an orthogonal system on a surface form a normal congruence, then the centers of geodesic curvature of a single infinity of orthogonal systems of curves lie on the lines of this congruence and corresponding curves of any two of these systems intersect under a constant angle.*

Subtracting (26) from (23) we obtain

$$h_u\left(\phi_v + r_1 + \frac{\eta_1 \cos\phi}{h}\right) = h_v\left(\phi_u + r - \frac{\xi \sin\phi}{h}\right). \tag{27}$$

From (24), (25) and (27) and from (21), (22) and (27) it follows that

$$h_u\,(\phi_v - \theta_v) - h_v\,(\phi_u - \theta_u) = 0, \tag{28}$$

$$h_u\, t_v - h_v\, t_u = 0. \tag{29}$$

THEOREM 10. *The three quantities h, t and $\phi - \theta$ are functions of one another.*

If in (21) and (22) h is assumed to be a function of t, the vanishing of the Jacobian $\dfrac{\partial\,(h, t)}{\partial\,(u, v)}$ is equivalent to

$$\begin{vmatrix} \xi \sin\phi - h\,\phi_u - r\;\; h, & h_u \\ -\eta_1 \cos\phi - h\,\phi_v - r_1\, h, & h_v \end{vmatrix} = 0.$$

This equation reduces to (26) and hence, together with Theorem 10, proves the following, which may be used as a second definition of associated normal congruences:

THEOREM 11. *The necessary and sufficient condition that a normal congruence whose lines lie in the tangent planes of a surface be an associated normal congruence is that h be a function of t, or that h or t be a constant, or that both be constants, and that ϕ be a solution of* (26).

§ 7. *Verification and Extension of the Results in the Preceding Section.*

Let us now verify formula (29) by means of the formulæ obtained in the proof of Caronnet's theorem. In that proof t was measured from $(-R_2, 0, 0)$ to any normal surface of the congruence and was given by

$$d\,t = \frac{R_2\, d\, R_2}{\sqrt{R_1^2 + R_2^2}}.$$

If the distance is to be measured from P_1, to t must be added the distance from $(-R_2, 0, 0)$ to P_1. The result of this substitution is

$$d\,t = \frac{R_1\, R_2}{(R_1^2 + R_2^2)^{3/2}}\,(R_2\, d\, R_1 - R_1\, d\, R_2).$$

Since h is the length $M P_1$,

$$\left.\begin{aligned} h &= \frac{R_1\, R_2}{\sqrt{R_1^2 + R_2^2}}, \qquad d\,h = \frac{R_2^3\, d\, R_1 + R_1^3\, d\, R_2}{(R_1^2 + R_2^2)^{3/2}}, \\ \frac{\partial\,(h\,t)}{\partial\,(u\,v)} &= \frac{R_1^2\, R_2^2}{(R_1^2 + R_2^2)^2}\,(R_{1v}\, R_{2u} - R_{1u}\, R_{2v}). \end{aligned}\right\} \tag{30}$$

But R_2 is a function of R_1, or either R_1 or R_2 is a constant or both are constants; hence the Jacobian is equal to zero and the condition that t be a function of h or that either or both be constants is satisfied.

In consequence of Theorem 9 and (30) it follows that

$$h = \frac{R_1\, R_2}{\sqrt{R_1^2 + R_2^2}} = \frac{R_1'\, R_2'}{\sqrt{R_1'^2 + R_2'^2}}, \tag{31}$$

where R_1' and R_2' are the radii of geodesic curvature of another orthogonal system of curves whose centers of geodesic curvature lie on the lines of the same associated normal congruence. Since R_2 is a function of R_1 and R_2' is a function of R_1', it is necessarily true from (31) that R_1 is a function of R_1'. But since the curves which have radii of geodesic curvature R_1 and those which have radii of geodesic curvature R_1' intersect under a constant angle, by Theorem 9, we have

THEOREM 12. *There exists on any surface a double infinity of systems of curves whose geodesic radii are functions of one another and the curves of whose families intersect under a constant angle. These systems determine the same associated normal congruences.*

It becomes necessary now to prove that every system of this kind determines a normal congruence and that the congruence so determined is identical with one determined by an orthogonal system. Assuming that $v =$ const. and $u =$ const. have geodesic radii R_1 and R_2 respectively and intersect under a constant angle ω, and that the x-axis of an associated moving trihedral is tangent to $v =$ const., the following relations are obtained easily from the formulæ of the moving trihedral and the formulæ of the geodesic curvature of the parametric curves:

$$E = \xi^2, \qquad F = \xi\,\xi_1, \qquad G = \frac{\xi_1^2}{\cos^2\omega} = \frac{\eta_1^2}{\sin^2\omega}, \qquad \xi_1 \sin\omega = \eta_1 \cos\omega,$$

$$r = \frac{\xi_{1u} - \xi_v}{\eta_1}, \qquad r_1 = \frac{\eta_{1u} - \xi_v \sin\omega\cos\omega}{\sin^2\omega},$$

$$R_1 = \xi/r, \qquad R_2 = \xi_1/(r_1 \cos\omega) = \eta_1/(r_1 \sin\omega).$$

The coordinates of any point P on the line of the congruence are

$$\left(\frac{-\sin\omega\, R_2\, t}{\sqrt{R_1^2 + R_2^2 - 2 R_1 R_2 \cos\omega}}, \quad R_1 - \frac{(R_1 - R_2\cos\omega)\, t}{\sqrt{R_1^2 + R_2^2 - 2 R_1 R_2 \cos\omega}}, \ 0\right),$$

where t is the distance from $(0, R, 0)$ to P. The direction cosines of the line are proportional to

$$R_2 \sin\omega, \quad R_1 - R_2\cos\omega, \quad 0.$$

The condition that P generate an orthogonal surface of the congruence reduces to

$$d\,t\,\sqrt{R_1^2 + R_2^2 - 2 R_1 R_2 \cos\omega} = (R_1 - R_2\cos\omega)\, d\,R_1.$$

By a course of reasoning the same as that employed in § 1, this shows that R_1 is a function of R_2, R_1 or R_2 is a constant or both are constants.

If in § 6 the equations of the system of curves had been

$$\frac{d\,v}{d\,u} = \frac{\xi}{\eta_1}\tan\theta, \qquad \frac{d\,v}{d\,u} = \frac{\xi}{\eta_1}\tan(\theta + \omega),$$

and the conditions that their centers of geodesic curvature should lie on the line of the congruence had been imposed, equations (24) and (25) would again have been obtained. Hence the problem of determining these new systems is the same as that of determining the orthogonal systems.

THEOREM 13. *The necessary and sufficient condition that the lines joining corresponding centers of geodesic curvature of a system whose curves intersect under a constant angle shall form an associated normal congruence, is that the radii of geodesic curvature be functions of one another or that one radius be a constant or that both be constant.*

Theorem 13 gives a third definition of associated normal congruences. Together with the previous discussion this shows that the consideration of the orthogonal systems is equivalent to the consideration of these new systems. When a single orthogonal system determining an associated normal congruence is known, all systems of curves whose families intersect under a constant angle and determine the same normal congruence are known.

We will now verify formula (28). Since the slope of MP_1 is $\tan\phi$, the following equations are easily obtained:

$$\tan\phi = -\frac{R_2 - R_1\tan\theta}{R_1 + R_2\tan\theta}, \qquad \tan(\phi - \theta) = -R_2/R_1, \tag{32}$$

$$d(\phi - \theta) = \frac{R_2\,dR_1 - R_1\,dR_2}{R_1^2 + R_2^2} = \frac{\sqrt{R_1^2 + R_2^2}}{R_1 R_2}\,dt. \tag{33}$$

The curves intersect under right angles, and hence h depends only on R_1 and R_2 and is thus the same as in the previous discussion. These facts enable us to write the relation

$$\begin{vmatrix} \phi_u - \theta_u, & h_u \\ \phi_v - \theta_v, & h_v \end{vmatrix} = \frac{(R_1^2 + R_2^2)^{1/2}}{R_1 R_2}\begin{vmatrix} t_u & t_v \\ h_u & h_v \end{vmatrix} = 0,$$

and hence equation (28) is verified. From (33), when t equals a constant, $\phi - \theta$ equals a constant and conversely. From (32), when R_2 is equal to a constant times R_1, $\phi - \theta$ is equal to a constant and conversely, except when $1/R_1 = 0$. In the latter case $\phi - \theta$ is equal to zero and the curves defined by (5) are geodesics.

THEOREM 14. *A necessary and sufficient condition that P_1 generate an orthogonal surface of the associated normal congruence, is that the radius of geodesic curvature of the curves of one family of the orthogonal system be equal to a constant times the radius of geodesic curvature of the curves of the other family, or that one radius of geodesic curvature be infinite.*

§ 8. *Developables Corresponding to the Lines of Curvature.*

Suppose now that the lines of curvature are parametric and that equations (5) and (6) define an orthogonal system on the surface, such that if T is the geodesic curvature of (5), $f(T)$ is the geodesic curvature of (6). Under

these conditions we wish to find when the lines of curvature on the given surface correspond to the developables of the associated normal congruence. To do this, we will obtain the displacements of a point P on the line of the congruence, impose the conditions that P move tangent to this line, eliminate t to derive the equations of the developable surfaces, and finally equate the coefficients of $d\,u^2$ and $d\,v^2$ to zero. The developables will then be given by $d\,u\,d\,v = 0$, or in other words, they will correspond to the lines of curvature on the given surface. The system of equations obtained by this process is:

$$q\,(\cos\theta - f'\sin\theta)\,T_u = 0,$$
$$p_1\,(\sin\theta + f'\cos\theta)\,T_v = 0,$$

where f' is the derivative of $f(T)$ with respect to T. This system is equivalent to the following systems:

$$\left.\begin{aligned}\sin\theta + f'\cos\theta = 0,\\ \cos\theta - f'\sin\theta = 0,\end{aligned}\right\}(34),\quad \left.\begin{aligned}p_1 = 0,\\ q = 0,\end{aligned}\right\}(35),\quad \left.\begin{aligned}T_u = 0,\\ T_v = 0,\end{aligned}\right\}(36),$$

$$\left.\begin{aligned}T_v = 0,\\ \cos\theta = f'\sin\theta,\end{aligned}\right\}(37),\qquad \left.\begin{aligned}T_u = 0,\\ \sin\theta = -f'\cos\theta,\end{aligned}\right\}(38),$$

$$\left.\begin{aligned}p_1 = 0,\\ \cos\theta = f'\sin\theta,\end{aligned}\right\}(39),\qquad \left.\begin{aligned}q = 0,\\ \sin\theta = -f'\cos\theta,\end{aligned}\right\}(40),$$

$$\left.\begin{aligned}T_v = 0,\\ q = 0,\end{aligned}\right\}(41),\qquad \left.\begin{aligned}T_u = 0,\\ p_1 = 0,\end{aligned}\right\}(42).$$

The determinant of (34) is $1 + f'^2$; hence it expresses an impossible condition. Since the lines of curvature are parametric, system (35) defines a plane. System (36) states that the geodesic curvatures of both families of curves are constant. This has been fully discussed in §§ 2–4. Systems (37), (39) and (41) are similar respectively to (38), (40) and (42). Hence it is advisable to consider the first set only. Since, as may be easily shown, system (39) is satisfied only by developable surfaces of revolution, it is not desirable to treat it further. The only surfaces that satisfy condition (41) are the plane and developable surfaces of revolution; but as this statement is by no means evident, it is desirable to prove it. The lines of curvature being parametric, system (41) is equivalent to the two systems

$$T_v = 0,\ q = 0,\ p_1 = 0,\ r_v = r_{1u},\ r = -\frac{\xi_v}{\eta_1},\ r_1 = \frac{\eta_{1u}}{\xi}, \tag{43}$$

$$T_v = 0,\ q = 0,\ r = 0,\ p_{1u} = 0,\ r_{1u} = 0,\ \xi_v = 0,\ r_1 = \frac{\eta_{1u}}{\xi}. \tag{44}$$

Evidently a surface which satisfies equations (43) must be a plane. The linear element of a surface conditioned by equations (44) is reducible to the form $ds^2 = du^2 + (v_1 u + v_2)^2 dv^2$, where v_1 and v_2 are functions of v alone. Since q and the total curvature of this quadratic form are zero, the curves $v = \text{const.}$ must be tangents to the edge of regression. If $v_2 = 0$ or $v_1 = 0$ or $v_1/v_2 = \text{const.}$, the surface is a right circular cone or cylinder, since it is then a developable surface of revolution on which the lines of curvature are parametric. In general, T and $f(T)$, which are functions of u alone, and v_1 and v_2 must be such quantities that

$$T \xi \eta_1 = \frac{\partial}{\partial u}(\eta_1 \sin\theta) - \frac{\partial}{\partial v}(\xi \cos\theta), \tag{45}$$

$$f(T) \xi \eta_1 = \frac{\partial}{\partial u}(\eta_1 \cos\theta) + \frac{\partial}{\partial v}(\xi \sin\theta) \tag{46}$$

possess solutions θ. Substituting for ξ and η_1 their values given by the linear element, and solving for θ_u and θ_v, we obtain

$$\theta_u = R\cos\theta - f\sin\theta,$$

$$\theta_v = -v_1 + (v_1 u + v_2)(R\sin\theta + f\cos\theta).$$

The condition of integrability of these equations is

$$T^2 + f^2 + T_u \sin\theta + f_u \cos\theta = 0.$$

By virtue of the identity $\sin^2\theta + \cos^2\theta = 1$, this equation shows that θ must be a function of u alone. Further, since θ is a function of u alone, $\dfrac{v_1}{v_1 u + v_2}$ is a function of u alone. Therefore $v_2/v_1 = \text{const.}$, and consequently the surface is a developable surface of revolution.

System (37), in which R, f and θ must be functions of u alone, will now be investigated at length. Since θ is a function of u alone, equations (45) and (46) are equivalent to

$$\frac{\eta_{1u}}{\xi \eta_1} = (T\sin\theta + f\cos\theta), \tag{47}$$

$$\eta_1 \theta_u = \xi_v + \xi\eta_1 (T\cos\theta - f\sin\theta). \tag{48}$$

To these must be added the identity

$$\sin^2\theta + \cos^2\theta = 1, \tag{49}$$

and from system (37) the equation

$$f_u \sin\theta = T_u \cos\theta. \tag{50}$$

From (48) and (49) we have

$$\sin\theta = \pm \frac{T_u}{\sqrt{T_u^2 + f_u^2}}, \tag{51}$$

$$\cos\theta = \pm \frac{f_u}{\sqrt{T_u^2 + f_u^2}}, \tag{52}$$

where the upper signs are to be taken together. According to equation (47) the geodesic curvature of $u = \text{const.}$ is a function of u alone. From (51) and (52) by differentiation we have

$$\theta_u = \frac{f_u T_{uu} - T_u f_{uu}}{T_u^2 + f_u^2}, \tag{53}$$

and from (53) and (48),

$$r = \frac{T_u f_{uu} - f_u T_{uu}}{T_u^2 + f_u^2} \pm \xi \frac{T f_u - f T_u}{\sqrt{T_u^2 + f_u^2}}. \tag{54}$$

By (47) the coordinates, with respect to the moving trihedral, of the centers of geodesic curvature of $u = \text{const.}$ are

$$\left(\mp \frac{\sqrt{T_u^2 + f_u^2}}{T T_u + f f_u},\ 0,\ 0\right), \tag{55}$$

and by use of (51) and (52) these are identical with the coordinates of the point of intersection of the line of the congruence and the x-axis. Therefore the center of geodesic curvature of $u = \text{const.}$ lies on the line of the congruence. The displacements of the point (55) are:

$$\delta x = \mp d\left(\frac{\sqrt{T_u^2 + f_u^2}}{T T_u + f f_u}\right) + \xi\, du,$$

$$\delta y = \mp \frac{1}{T T_u + f f_u}\left[\frac{T_u f_{uu} - f_u T_{uu}}{\sqrt{T_u^2 + f_u^2}} + \xi\,(T f_u - f T_u)\right] du,$$

$$\delta z = \pm q \frac{\sqrt{T_u^2 + f_u^2}}{T T_u + f f_u}\, du.$$

Since none of these displacements involve dv, the point (55) must generate a curve as M moves over the surface. It follows also that ξ and q must be functions of u alone. Hence without loss of generality ξ can be taken equal to unity. This then proves with (47) that η_1 is a function of u alone. Therefore since the linear element of the surface is of the form $ds^2 = du^2 + \eta_1^2\, dv^2$, and the lines of curvature are parametric, the surface must be a surface of revolution.

Since ξ is unity and r is zero, from (54) it follows that

$$\pm \frac{T f_v - f T_u}{\sqrt{T_u^2 + f_u^2}} = \frac{f_u T_{uu} - T_u f_{uu}}{T_u^2 + f_u^2}, \tag{56}$$

and from (46) that

$$\frac{\eta_{1u}}{\eta_1} = \pm \frac{T T_u + f f_u}{\sqrt{T_u^2 + f_u^2}}. \tag{57}$$

In equations (56) and (57) there are three functions of u. If it is possible to choose $\eta_1(u)$ arbitrarily and determine functions T and f that satisfy (56) and (57), the preceding discussion will hold true for *any* surface of revolution. Let then η_1 be a known analytic function of u and denote by U the function $\frac{\eta_{1u}}{\eta_1}$. In order to further simplify the notation, use only upper signs and let dT/du be μ and df/du be λ. Then the equations to be considered are:

$$U\sqrt{\lambda^2 + \mu^2} = T\mu + f\lambda, \tag{58}$$

$$(T\mu + f\lambda)(T\lambda - f\mu) = U\left(\lambda \frac{d\mu}{du} - \mu \frac{d\lambda}{du}\right). \tag{59}$$

Solving (58) for μ we have

$$\mu = \lambda \frac{-Tf \pm U\sqrt{T^2 + f^2 - U^2}}{T^2 - U^2}. \tag{60}$$

Both of these values of μ satisfy (58). For convenience choose one of them and let μ equal $\lambda\phi(T, f, u)$. By substituting the values of μ and $d\mu/du$ in equation (59), it is found that the coefficient of $d\lambda/du$ is identically zero and that (59) reduces to

$$\lambda^2\left[\lambda U\left(\phi\frac{\partial\phi}{\partial T} + \frac{\partial\phi}{\partial f}\right) + U\frac{\partial\phi}{\partial u} - (T - f\phi)(T\phi + f)\right] = 0. \tag{61}$$

If λ equals zero, $f(T)$ degenerates into a constant. By formula (52) this means that θ equals $\pi/2$ or $3\pi/2$, and hence that the curves whose equation is (5) are the parallels of the surface of revolution, and those whose equation is (6) are the meridians. The constant f must then be zero and T must be U, which is equal to $\frac{\eta_{1u}}{\eta_1}$. These results satisfy equations (58) and (59).

Let us now consider the other value of λ. Since neither of the coefficients $U\left(\phi\frac{\partial\phi}{\partial T} + \frac{\partial\phi}{\partial f}\right)$ and $U\frac{\partial\phi}{\partial u} - (T - f\phi)(T\phi + f)$ is identically zero, equation (61) will in general give a definite value of λ which is not zero or infinite.

If we denote these values of λ and μ by the following analytic functions of T, f and u,

$$\mu = \Phi\,(T, f, u), \qquad \lambda = \Psi\,(T, f, u),$$

equations (58) and (59) will become equivalent to

$$\frac{dT}{du} = \Phi\,(T, f, u), \qquad \frac{df}{du} = \Psi\,(T, f, u).$$

Hence we can say that for any given $\eta_1(u)$ there exist functions T and f, which satisfy equations (58) and (59). When T and f are determined, the angle θ can be found by means of equation (51) and (52). In this discussion we have used the upper signs. The same results would follow if we had used the lower signs. As no difficulty arises in the determination of p, q, r and r_1, we have as the result of the investigation in this section the theorem: *

THEOREM 15. *The necessary and sufficient condition that there exist associated normal congruences whose developables correspond to the lines of curvature on the given surface is that the surface be a surface of revolution or a pseudospherical surface.*

If in the preceding discussion of surfaces of revolution $f\,(T)$ is equal to a constant times T, the point (55), which is the center of geodesic curvature of the parallels of the surface of revolution, is the point P_1. This point generates an orthogonal surface of the congruence and the congruence so obtained is the same as that determined by the parallels and meridians. Conversely, if it is assumed that the point (55) generates an orthogonal surface of the congruence, by means of (54) we find that f must equal a constant times T.

For any parallel the totality of lines of the congruence will envelope a right circular cone. Hence the orthogonal surfaces will be surfaces of revolution. When f is equal to a constant times T, the lines corresponding to any parallel will be perpendicular to the axis and also to a tangent to the corresponding meridian. The orthogonal surfaces of the congruence will, in this case, be right circular cylinders.

§ 9. *The Geodesic Curvature of One Family only of an Orthogonal System on a Surface is a Constant. Surfaces on Which One Family of Lines of Curvature Have Constant Geodesic Curvature.*

It has already been established that the center of constant geodesic curvature generates an orthogonal surface of the associated normal congruence,

* The total curvature of the surface of revolution is

$$K = -(T^2 + f^2) \mp (T_u^2 + f_u^2)^{1/2}.$$

that the point P_1 can not lie on an orthogonal surface of this congruence, and that, unless the orthogonal system is made up of the lines of curvature of a surface of revolution, the developables of the congruence can not correspond to the lines of curvature on the given surface. For convenience of notation let c equal $1/a$, where a is the geodesic curvature of the curves $u=$ const. of the orthogonal parametric system. Then with respect to the moving trihedral, chosen as usnal, the coordinates of the center of constant geodesic curvature are $(-c, 0, 0)$. The displacements of this point are

$$\delta x = \xi\, du, \quad \delta y = -c\, r\, du, \quad \delta z = (q\, du + q_1\, dv)\, c,$$

and consequently the linear element of the orthogonal surface generated by it is

$$ds^2 = (\xi^2 + c^2 r^2 + c^2 q^2)\, du^2 + 2 c^2 q\, q_1\, du\, dv + c^2 q_1^2\, dv^2. \tag{62}$$

If the parametric system is to be orthogonal on this surface, q must equal zero. This means that the curves $v=$ const. on the given surface are asymptotic lines. These asymptotic lines can not be straight lines, as then both geodesic curvatures would be constant and the surface would be pseudospherical.

Since

$$\eta = \xi_1 = 0, \qquad p_1\, \eta - q_1\, \xi = p\, \eta_1 - q\, \xi_1,$$

it follows that if q_1 equals zero, p must equal zero also. This condition will soon be discussed.

If the curves $u=$ const. are to be lines of curvature, the conditions imposed on the surface are

$$\eta_{1u} = a\,\xi\,\eta_1, \;\; r = -\xi_v/\eta_1, \;\; r_1 = \eta_{1u}/\xi, \;\; p_{1u} = -q\, r_1, \;\; q_v = r\, p_1, \;\; r_v - r_{1u} = -p_1 q.$$

These are equivalent to

$$r_1 = a\,\eta_1, \tag{63}$$
$$\eta_{1u} = a\,\xi\,\eta_1, \tag{64}$$
$$p_{1u} = -a\, q\, \eta_1, \tag{65}$$
$$q_v = r\, p_1, \tag{66}$$
$$r_v = a^2\,\xi\,\eta_1 - p_1\, q, \tag{67}$$
$$\xi_v = -r\,\eta_1. \tag{68}$$

Multiplying (66) by q and (67) by r, and adding, gives

$$q\, q_v + r\, r_v + a^2\,\xi\,\xi_v = 0,$$

or after integration,

$$q^2 + r^2 + a^2\,\xi^2 = U^2,$$

where U is a function of u alone. By the transformation $du_1 = U\, du$, this

becomes $r'^2 + q'^2 + a^2 \xi'^2 = 1$; hence without loss of generality U may be taken as unity. Let then U be unity and let ξ, q and r be represented by $\xi = c \sin\theta$, $q = \sin\phi \cos\theta$, $r = \cos\phi \cos\theta$.

From the conditions above we also have

$$\eta_1 = -\frac{c\,\theta_v}{\cos\theta}, \qquad r_1 = -\frac{\theta_v}{\cos\phi}, \qquad p_1 = \phi_v - \tan\phi \tan\theta\,\theta_v.$$

The fundamental quantities are now expressed in terms of the new variables ϕ and θ, and equations (64) and (65) are left to determine these. Substituting in (65) and (64) respectively gives

$$\begin{aligned} &\phi_{uv} - \tan\phi \tan\theta\,\theta_{uv} - \tan\phi \sec^2\theta\,\theta_u\,\theta_v \\ &\qquad - \sec^2\phi \tan\theta\,\phi_u\,\theta_v - \tan\phi \cos\theta\,\theta_v = 0, \end{aligned} \tag{69}$$

$$\theta_{uv} + \tan\phi\,\phi_u\,\theta_v - \sin\theta\,\theta_v = 0. \tag{70}$$

Eliminating θ_{uv} from (69) by means of (70) we have

$$\phi_{uv} - \tan\theta\,\theta_v\,\phi_u - \tan\phi \sec^2\theta\,\theta_u\,\theta_v - \tan\phi \sec\theta\,\theta_v = 0. \tag{71}$$

Any pair of solutions ϕ and θ of equations (70) and (71) except those for which θ_v is zero will determine ξ, η_1, r, r_1, p_1 and q of a surface on which the lines of curvature $u=$const. have constant geodesic curvature a.

It may now be shown that the curves $u =$ const. are spherical lines of curvature. The normal curvature of $u =$ const., denoted by $1/\rho_2$, is equal to p_1/η_1. Substituting the values of p_1 and η_1 just found, we have

$$-\rho_2 = \frac{a\,\theta_v}{\cos\phi\,\phi_v - \sin\phi \tan\theta\,\theta_v}.$$

The curves $u=$const. are spherical if

$$-\rho_2 = U_1 + \frac{U_2}{\sqrt{\mathcal{E}\mathcal{G}}} \frac{\partial \sqrt{\mathcal{G}}}{du},$$

where U_1 and U_2 are functions of u alone. Since in our notation $\mathcal{E} = q^2$ and $\mathcal{G} = p_1^2$, it follows that

$$\frac{1}{\sqrt{\mathcal{E}\mathcal{G}}} \frac{\partial \sqrt{\mathcal{G}}}{\partial u} = \frac{\phi_{uv} - \sec^2\phi \tan\theta\,\phi_u\,\theta_v - \tan\phi \sec^2\theta\,\theta_u\,\theta_v - \tan\phi \tan\theta\,\theta_{uv}}{\sin\phi \cos\theta\,\phi_v - \sin\phi \tan\phi \sin\theta\,\theta_v}.$$

By using (69) this becomes

$$= \frac{\tan\phi \cos\theta\, \theta_v}{\sin\phi \cos\theta\, \phi_v - \sin\phi \tan\phi \sin\theta\, \theta_v}$$

$$= \frac{\theta_v}{\cos\phi\, \phi_v - \sin\phi \tan\theta\, \theta_v}$$

$$= \frac{-\rho_2}{a}.$$

This proves that the curves $u =$ const. are spherical.

Since the lines of curvature are parametric, q_1 equals zero and the linear element (62) becomes $d s^2 = c^2 d u^2$. Hence the orthogonal surface of the congruence generated by the center of constant geodesic curvature is a curve and therefore a focal surface. It may now be geometrically shown that the curves $u =$ const. are spherical. Denoting the centers of geodesic curvature of $u =$ const. and $v =$ const. by A and B respectively, and the line of the congruence by AB, we see that for any curve $u =$ const., A remains fixed as v varies and M, the point of tangency of the tangent planes, must move so that it remains at a constant distance c from A. Since the tangent to $v =$ const. passes through A, the sphere on which $u =$ const. lies cuts the surface at right angles. Any surface orthogonal to the congruence AB is generated by a point P at a constant distance from A on AB. When $u =$ const., as v varies, P_1 generates a circle since it lies at a constant distance from A on AB, which is perpendicular to the curve generated by A. Hence any orthogonal surface of the associated normal congruence is the locus of spheres of constant radius whose centers are on the curve generated by A. These circles are lines of curvature.

The lines joining A and the corresponding center N of normal curvature of $v =$ const. form a normal congruence.* We desire to show that the congruence generated by AB and AN are the same. Any point P on AN has the coordinates

$$\left(-c + \frac{c\,t}{\sqrt{c^2 + \rho_1^2}},\ 0,\ \frac{\rho_1\, t}{\sqrt{c^2 + \rho_1^2}}\right),$$

where t is the distance from a to P and ρ_1 is the radius of normal curvature of $v =$ const. The direction cosines of AN are proportional to c, 0 and ρ_1. If the condition that P move perpendicular to AN for all displacements on

* *Comptes Rendus*, Vol. CXV, pp. 589–592.

the surface in the neighborhood of M is expressed, it is found that the resulting equation is satisfied by $t = \text{const.}$ Hence the curve generated by A is orthogonal to both congruences. All the lines AB and AN corresponding to any curve $u = \text{const.}$ thus lie in the same plane. These two congruences, which are differently defined, may be considered as the same, since any line of either is a line of the other. The common perpendicular to any two lines AB and AN is the tangent at A of the curve A generates. If ω is the angle between these two lines, it is easily shown that

$$\cos\omega = \frac{c^2}{\sqrt{(c^2 + R^2)\,(c^2 + \rho_1^2)}}.$$

Let us now obtain the equations of a surface on which $u = \text{const.}$ are lines of curvature and have constant geodesic curvature a. In order to do this let $x_0(u)$, $y_0(u)$ and $z_0(u)$ define the locus of centers of spheres of radius c, and $\alpha(u, v)$, $\beta(u, v)$ and $\gamma(u, v)$ be the direction cosines of the radii of the spherical curves $u = \text{const.}$ Then the equations of the desired surface are

$$x = x_0 + c\,\alpha, \qquad y = y_0 + c\,\beta, \qquad z = z_0 + c\,\gamma. \tag{72}$$

Since these radii are tangent to $v = \text{const.}$, we have

$$x_u = x_{0u} + c\,\alpha_u = r\,\alpha, \tag{73}$$

$$y_u = y_{0u} + c\,\beta_u = r\,\beta, \tag{74}$$

$$z_u = z_{0u} + c\,\gamma_u = r\,\gamma, \tag{75}$$

where r is a factor of proportionality. To these must be added

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \tag{76}$$

Multiplying equations (73), (74) and (75) by α, β and γ respectively, and adding, we have $r = \Sigma\, x_{0u}\,\alpha$. Hence, if u measures the arc of the curve of centers and ω is the angle between a tangent to $v = \text{const.}$ and the corresponding tangent to this curve, it follows that $r = \cos\omega$. It is necessary to prove that the equations (73), (74), (75) and (76) are consistent. To do so eliminate γ from (73), (74) and (75) by means of (76), and then, in the equation resulting from the last elimination, substitute the values of α_u and β_u from the equations obtained by the first two eliminations. The equality so obtained is an identity. This shows that the three equations in α and β are not independent and that these quantities may be determined by the following:

$$c\,\alpha_u = x_{0u}(\alpha^2 - 1) + y_{0u}\,\alpha\,\beta + z_{0u}\,\alpha\sqrt{1 - \alpha^2 - \beta^2},$$

$$c\,\beta_u = x_{0u}\,\alpha\,\beta + y_{0u}(\beta^2 - 1) + z_{0u}\,\beta\sqrt{1 - \alpha^2 - \beta^2}.$$

Therefore for any given set of initial coordinates α_0, β_0, u_0 and v_0 for which $\alpha_0^2 + \beta_0^2 \leqq 1$ and x_0, y_0 and z_0 are analytic, there exist solutions $\alpha(u, v)$ and $\beta(u, v)$ which, for $u = u_0$, reduce to functions $\phi_1(v)$ and $\phi_2(v)$ respectively, and for $u = u_0$ and $v = v_0$ reduce to α_0 and β_0. After α and β have been determined, γ is known and hence also the equation of the surface.

APPENDIX.

Conformal Representation on the Plane of Systems of Curves of Constant Geodesic Curvature that lie on a Pseudospherical Surface.

If the linear element of a pseudospherical surface is

$$d s^2 = c^2 (d u^2 + \varepsilon^{2u} d v^2),$$

Darboux has shown, by means of the transformation*

$$v = x, \qquad \varepsilon^{-u} = y,$$

that curves on the surface of geodesic curvatures a and b may be represented conformally on the upper half plane by the circles

$$(x - h)^2 + (y - l)^2 = \frac{l^2 c^2}{a^2}, \tag{I}$$

$$(x - m)^2 + (y - n)^2 = \frac{n^2 c^2}{b^2}. \tag{II}$$

Since systems of curves whose two families have geodesic curvatures a and b and intersect under a constant angle ω determine associated normal congruences, we wish to find the corresponding systems of circles in the plane that represent them conformally. If we suppose that the circles (I) and (II) intersect under an angle ω, it follows that

$$(h - m)^2 + (l - n)^2 = c^2 \left(\frac{l^2}{a^2} + \frac{n^2}{b^2} - 2 \frac{l n}{a b} \cos \omega\right). \tag{III}$$

Previously it has been shown that

$$c^2 = \frac{a^2 + b^2 - 2 a b \cos \omega}{\sin^2 \omega}. \tag{IV}$$

This relation enables us to write equation (III) thus:

$$\left[\sin \omega (h - m) + l\left(\cos \omega - \frac{b}{a}\right) + n\left(\cos \omega - \frac{a}{b}\right)\right]\left[\sin \omega (h - m) - l\left(\cos \omega - \frac{b}{a}\right) - n\left(\cos \omega - \frac{a}{b}\right)\right] = 0. \tag{V}$$

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* Darboux, "Leçons," Vol. III, Ch. XI.

There is a single infinity of circles in each family of the system, and hence we will suppose that the pairs h and l and m and n are functions respectively of the parametrics s and t. Considering the first factor only of (V), differentiation with respect to s and t gives the two equations

$$\sin\omega\, h_s + \left(\cos\omega - \frac{b}{a}\right) l_s = 0,$$

$$\sin\omega\, m_t - \left(\cos\omega - \frac{a}{b}\right) n_t = 0.$$

Since $\frac{l_s}{h_s}$ and $\frac{n_t}{m_t}$ are the tangents of the angles the directions of motion of the centers (h, l) and (m, n) make with the x-axis, and are constants, the curves of centers must necessarily be straight lines. From the first factor of (V) we obtain, then, that the equations of the lines of centers must be of the form

$$h \sin\omega + l\left(\cos\omega - \frac{b}{a}\right) + K = 0, \tag{VI}$$

$$m \sin\omega - n\left(\cos\omega - \frac{a}{b}\right) + K = 0, \tag{VII}$$

where K is the constant that determines the particular system for given a and b. These lines intersect at the point $\left(-\frac{K}{\sin\omega}, 0\right)$ under an angle ω.

As corresponding curves of any two orthogonal systems, of the kind under consideration, that determine the same associated normal congruence intersect under a constant angle, we desire to prove that the same is true for corresponding circles of two orthogonal systems of circles whose lines of centers pass through the same point on the x-axis. Thus, for the present, assume that ω is a right angle and that we select two sets of values a_1 and b_1 and a_2 and b_2 for a and b. If we take the same value of K, say K_1, for both systems, the lines of centers of the circles will intersect on the axis of x at the same point, and the lines of centers of the circles corresponding to the curves of geodesic curvature a_1 and a_2 will intersect under some angle ω'. If we can show that a_1, a_2 and ω' satisfy a relation of the form (III), where

$$a_i^2 + b_i^2 = c^2, \qquad i = 1, 2,$$

the system of circles so obtained will correspond conformally to a system of curves on the surface which intersect under a constant ω', and hence will them-

selves intersect under the same angle. From the equations of the lines of centers it follows that

$$\cos \omega' = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}},$$

$$\therefore c^2 = \frac{a_1^2 + a_2^2 - 2 a_1 a_2 \cos \omega'}{\sin^2 \omega'}.$$

Therefore the desired result is proved.

An examination of the pairs of equations (I) and (VI) and (II) and (VII) yields that all the circles of each family are tangent to each other at the point $\left(-\frac{K}{\sin \omega}, 0\right)$, that there are only two circles of infinite radius in the system and that through any point in the upper half plane there pass one and only one pair of circles of the system.

THEOREM. *That part of all systems of circles, lying in the upper half plane, which satisfies conditions* (I), (II), (V), (VI) *and* (VII), *where a particular value for* $\frac{K}{\sin \omega}$ *has been chosen, represents conformally all systems of curves of geodesic curvatures a and b on a pseudospherical surface, which determine the same associated normal congruence.*

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